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PhD Thesis Summary

Positive linear operators, inequalities, classes of convex functions

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Table of contents

Introduction

Approximation Theory is multifaceted theory. Positive linear operators play an important role in this realm. They are used in order to obtain qualitative results expressed by the convergence of approximating functions toward a prescribed function. The rate of convergence in such a process is described in terms of specific inequalities and involving moduli of continuity. Properties of the approximated functions can be used in order to obtain better rates of convergence. In this sense convex functions and generalized convex functions can be approximated with a higher speed if we choose suitable classes of positive linear operators.

On the other hand the degree of smoothness of a function influences the degree of approximation. In this context the Voronovskaja type results play a significant role. The behavior of a positive linear operator L_n with respect to the polynomials is important for computing its moments and central moments which are essential in establishing Voronovskaja type results. Therefore, the eigenstructure of L_n provides important information concerning the images of polynomials under L_n. In particular the kernel of L_n is an important subject of study.

Starting with a given sequence of positive linear operators, some modifications could have better approximation properties or better shape preserving properties. In this sense Kantorovich type modifications are often used. The iterates of positive linear operators are important in Approximation Theory as well as in Ergodic Theory. The convergence of such iterates can be studied in some instances by using results and algorithms from linear algebra.

Classical positive linear operators usually preserve the affine functions. In the last years special attention has been paid to the construction of new sequences preserving one or two prescribed functions. This enlarged the family of functions which can be approximated using positive linear operators.

The present thesis is concerned with topics related to the above context. This Introduction is followed by 4 chapters and a list of references.

Chapter 1, "Voronovskaja type results for sequence of operators", targets two important aspects. Firstly, we show Voronovskaya type formulas that can be "differentiated" (by changing the order of limit and differentiation), or which are associated with operators fixing two given functions. Secondly, we get new results regarding some special sequences of positive linear operators previously investigated in literature; in particular their Voronovskaya formula can be as well "differentiated". Appell polynomials are involved in the study of the moments of these operators.

Section 1.1 is dedicated to the "differentiation" of a Voronovskaya formula. The main outcome can be presented as: if the Voronovskaya formula for the sequence $(L_n)_{n\geq 1}$ can be "differentiated", then the formula associated with the operators modified in the sense of [14] can be also "differentiated".

Within Section 1.2 we talk about operators preserving two monomials or, more generally, two powers of a given bijection that are obtained by modifying classical ones.

Moreover, approximation properties are described in correlation to their associated Voronovskaya operators. Concretely, we start with a classical sequence of positive linear operators (Bernstein, Meyer-König and Zeller) and modify it to obtain a new sequence of operators which preserve the monomials x^i and x^j. For such modified sequences we establish the Voronovskaja formulas and compare the magnitude of the Voronovskaja operators. This gives information about the quality of approximation. We find intervals on which one sequence provides approximation better than other sequences.

When a positive linear operator fixes two functions φ, ψ, a subject of interest is to consider its behaviour with respect to functions which are convex relative to $\{\varphi, \psi\}$ (see [14]). The study of their behaviour with respect to the coresponding generalized convex functions could be a topic of further work. In this sense the Voronovskaja limit of a sequence of operators gives important information concerning the behaviour of the operators in relation with suitable classes of convex functions.

In [75] and [20] the authors introduced a sequence of positive linear operators L_n described in terms of divided differences or, equivalently, in different terms. Across Section 1.3 we show that these operators possess the property of commutativity with the ordinary differential operator; consequently, they are invariant under the Kantorovich type modification. We give explicit expressions for the images of exponentials and trigonometric functions under L_n. Furthermore, we show that for each fixed n the images of the monomials under L_n form a sequence of Appell polynomials. This leads to some algebraic identities involving the Stirling numbers of second kind. In addition, results showing that the central moments of L_n are constant functions for which we provide estimates, are mentioned.

The final section of this chapter features the topic of recently introduced Rathore type operators $R_{n,c}$. Concerning these operators we provide Voronovskaya type results and a comparison with the extended Szász-Mirakjan operators. Observing that the condition $f^{(3)} \geq 0$ expresses a property of generalized convexity, we intend to extend the result of the type mentioned in Remark 1.9 to other classes of generalized convex functions.

Chapter 2 is entitled "Positive linear operators: eigenstructure and applications to difference equations". In this chapter we consider the operators given in Section 1.3, acting on $C(\mathbb{R})$. Section 2.1 is devoted to the operators L_n restricted to polynomials. In order to describe the eigenstructure of operators we investigate the matrices of $L_n : \Pi_{2k} \to \Pi_{2k}$, respectively $L_n : \Pi_{2k+1} \to \Pi_{2k+1}$, with respect to the monomial bases. The kernel of the operator L_n acting on $C(\mathbb{R})$ is subject of study in Section 2.2. This kernel is related to the set of solutions of a difference equation. Given $T > 0$ and $p \in \Pi_m$, we provide three algorithms in order to find $q \in \Pi_{m+1}$ such that $q(x+T) - q(x) = p(x), x \in \mathbb{R}$. The algorithms are useful in inverting the operator $L_n : \Pi \to \Pi$ and, ultimately, in solving some difference equations. Section 2.3 deals with the eigenstructure of the Beta type operators from the general family depending on the parameters $\alpha > -1$, $\beta > -1$. We find the eigenvalues in explicite forms and give recurrence relations for the coefficients of the eigenpolynomials. The limiting case when $\alpha \to -1$, $\beta \to -1$ was considered in [39]. Some results from [39] are covered by our results from Section 2.3.

Chapter 3 intends to present instances of modified positive linear operators, their iterates and systems of linear equations. The cases of the Kantorovich modifications of linking operators and Stancu modifications of Bernstein operators are brought in this topic.

Section 3.1 deals with the Kantorovich modifications of linking operators and presents their study with respect to the limit of iterates and the invariant measure. In particular we give a new proof of [12, Theorem 5.1]. A concrete example, where the invariant measure is the Lebesgue measure, is presented at the end of the section.

In Section 3.2 we consider a family of Stancu operators, see [41], [40], and investigate the limit of the iterates of such an operator. In finding the limit of the iterates an essential step is the solution of a linear system of algebraic equations.

Section 3.3 is devoted to a modification of the sequence of Bernstein operators B_n on C[0,1], introduced by Schnabl [78] in order to investigate the global saturation of the sequence (B_n). We show that they can be obtained as a particular case of Stancu operators, and so the results from Section 3.2 can be applied. The operators C_n introduced by Schnabl do not preserve the constant function e_0. In fact, $C_n e_0 = \dfrac{n-1}{n} e_0$. Therefore, the Voronovskaja formula for the sequence C_n established by Schnabl contains in the right hand side not only f' and f'', but also the function f

(see (3.16)). We consider the operators $A_n = \frac{n}{n-1} C_n$, $n \geq 2$. The Voronovskaja formula for the sequence (A_n) is simpler (see (3.17)). The section ends with the result concerning the rate of convergence of the sequence (A_n).

As mentioned above, the limit of the iterates is determined by solving a system of linear equations. In this sense an algorithm $A(p)$, $p \in \mathbb{R}$ is presented in Section 3.4. $A(1)$ concides with the EMML algorithm (the Expectation-Maximization (EM) algorithm in order to compute Maximum Likelihood (ML) estimates; the same algorithm is used in the setting of restoration of astronomical images). $A(-1)$ is a version of the Image Space Reconstruction Algorithm (ISRA). At the end of the section the algorithm is illustrated with numerical experiments.

Usually for practical purposes, a sequence of positive linear operators is convergent to the identity operator. Under a specific modification, with a probabilistic flavour, a new sequence will converge towards an operator different from the identity, see, e.g., [13, 28]. In Section 3.5 we present results of this type involving a modification of Stancu operators $L_{n,\beta,\gamma}$.

Section 3.6 gives a hint on possible further work related to the above stuff.

Chapter 4 is entitled "Operators fixing exponential functions". Several papers can be found in the literature, dealing with positive linear operators which fix certain functions. A starting point was the paper written in 2003 by P.J. King [53] who constructed positive linear operators on $C[0,1]$ fixing the constant function 1 and the function x^2. Positive linear Bernstein-type operators fixing 1 and a given function τ with suitable properties were introduced and studied in [29]. Recently, operators fixing two exponential functions were constructed. Aral, Cardenas-Morales, Garrancho [22] considered a generalization of the classical Bernstein operators introduced by Morigi and Neamtu in 2000. Specifically, they focus on a sequence of operators that reproduce the exponential functions $\exp(\mu t)$ and $\exp(2\mu t)$, $\mu > 0$. Other papers devoted to operators fixing specific functions are [5], [11], [14], [22], [29], [36], [45].

We modify the Bernstein-Stancu operators such that the new operators $\tilde{S}_n$ preserve the functions $\exp_\mu$ and $\exp_\mu^2$, $\mu > 0$. This property is presented in Lemma 4.2 ii) and iii). A Voronovskaja type result for the newly introduced operators can be found in Theorem 4.1. Special classes of generalized monotone functions and convex functions are considered in this chapter. Shape preserving properties of the operators $\tilde{S}_n$ with respect to these classes of functions are described in Section 4.1. Section 4.2 is devoted to a comparison between the operators $\tilde{S}_n$ and the classical Bernstein-Stancu operators. For some functions we present graphical experiments showing that the operators $\tilde{S}_n$ provide a better approximation.

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